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On tensor products of representations of the non-standard q-deformed algebra $U'_q(\mathrm{so}_n)$

N Z Iorgov

Bogolyubov Institute for Theoretical Physics, Kiev 03143, Ukraine

E-mail: mmtpitp@bitp.kiev.ua

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Abstract

The tensor product of vector and arbitrary representations of the non-standard q-deformation $U'_q(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}_n)$ of Lie algebra so_n is defined. The Clebsch–Gordan coefficients of the tensor product of vector and arbitrary classical type representations of q-algebra $U'_q(\mathrm{so}_n)$ are found in an explicit form. Some important corollaries are considered. In particular, the Wigner–Eckart theorem for vector operators is proved.

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1. Introduction

For the last fifteen years, much attention by mathematicians and mathematical physicists has been paid to the subject of quantum algebra and quantum groups. Besides the standard deformation of Lie algebra proposed by Drinfeld [1] and Jimbo [2], other (non-standard) deformations are also under consideration. This paper deals with the deformation $U'_q(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}_n)$ proposed by Gavrilik and Klimyk [3]. Let us mention that the algebra $U'_q(\mathrm{so}_3)$ appeared earlier in [4].

As a matter of interest, the algebra $U'_q(\mathrm{so}_n)$ arose naturally as auxiliary algebra in deriving the algebra of observables in 2 + 1 quantum gravity with two-dimensional space of genus g, so that n depends on g, $n = 2g + 2$ [5–8].

As shown in [9], due to the existence of chain embeddings

$$U'_q(\mathrm{so}_n) \supset U'_q(\mathrm{so}_{n-1}) \supset \cdots \supset U'_q(\mathrm{so}_3) \supset U'_q(\mathrm{so}_2)$$

the algebra $U'_q(\mathrm{so}_n)$ admits a q-analogue of the Gel'fand–Tsetlin (GT) formalism for construction of finite-dimensional irreducible representations. In particular, the representations parametrizing by the highest weights of representations of the corresponding Lie algebra and reducing them in the limit $q \to 1$ were constructed. They are called classical type representations.

Since the algebra $U'_q(\mathrm{so}_n)$ is not a Hopf algebra, there is no natural way to introduce the notion of a tensor product of representations. But, as shown in [11–13], the algebra $U'_q(\mathrm{so}_n)$ is a subalgebra in the Drinfeld–Jimbo Hopf algebra $U_q(\mathrm{sl}_n)$. Moreover, it is possible to show that the algebra $U'_q(\mathrm{so}_n)$ is a $U_q(\mathrm{sl}_n)$-comodule algebra such that the coaction coincides with the comultiplication in $U_q(\mathrm{sl}_n)$ if one embeds $U'_q(\mathrm{so}_n)$ into $U_q(\mathrm{sl}_n)$. This comodule structure can be used to introduce the tensor product of vector and arbitrary representations T of $U'_q(\mathrm{so}_n)$ (which will be denoted by $T^{\otimes}$).

We obtain the decomposition of $T^{\otimes}$ into irreducible subrepresentations and find the corresponding Clebsch–Gordan coefficients (CGCs) in the case when T is an irreducible finite-dimensional representation of the classical type. It is shown that decomposition of $T^{\otimes}$ has the same form as in the case of Lie algebra so_n and the corresponding CGCs are q-deformations of their classical analogues [14, 15]. Taking into account this decomposition and embedding $U'_q(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$, we obtain some important corollaries.

It is well known that the Wigner–Eckart theorem for the tensor operators with respect to Lie algebra so_n (and, especially, so_3) is very important in physics. In this paper, we give a q-analogue for the case of vector operators.

Throughout we suppose that q is not a root of unity.

2. The q-deformed algebra $U'_q(\mathrm{so}_n)$

According to [3], the non-standard q-deformation $U'_q(\mathrm{so}_n)$ of the Lie algebra so_n is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ obeying the defining relations

$$\begin{aligned} &I_{j,j-1}^2 I_{j-1,j-2} + I_{j-1,j-2} I_{j,j-1}^2 - [2]\, I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2} \\ &I_{j-1,j-2}^2 I_{j,j-1} + I_{j,j-1} I_{j-1,j-2}^2 - [2]\, I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1} \\ &[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad |i-j| > 1 \end{aligned} \tag{1}$$

where $q + q^{-1} \equiv [2]$, $q \in \boldsymbol{C}$, $q \neq 0, \pm 1$. Along with the definition in terms of trilinear relations, we also give a 'bilinear' presentation. To this end, we introduce the generators

$$I_{k,l}^{\pm} \equiv [I_{l+1,l}, I_{k,l+1}^{\pm}]_{q^{\pm 1}} \qquad k > l+1 \quad l \geqslant 1 \quad k \leqslant n \tag{2}$$

where $[X, Y]_{q^{\pm 1}} \equiv q^{\pm 1/2} XY - q^{\mp 1/2} YX$ and $I_{k+1,k}^{+} \equiv I_{k+1,k}^{-} \equiv I_{k+1,k}$. Then (1) implies

$$\begin{aligned} &[I_{lm}^{\pm}, I_{kl}^{\pm}]_{q^{\pm 1}} = I_{km}^{\pm} \qquad [I_{kl}^{\pm}, I_{km}^{\pm}]_{q^{\pm 1}} = I_{lm}^{\pm} \qquad [I_{km}^{\pm}, I_{lm}^{\pm}]_{q^{\pm 1}} = I_{kl}^{\pm} \qquad \text{if} \quad k > l > m \\ &[I_{kl}^{\pm}, I_{mp}^{\pm}] = 0 \qquad \text{if} \quad k > l > m > p \quad \text{or} \quad k > m > p > l \\ &[I_{kl}^{\pm}, I_{mp}^{\pm}] = \pm (q - q^{-1}) (I_{lp}^{\pm} I_{km}^{\pm} - I_{kp}^{\pm} I_{ml}^{\pm}) \qquad \text{if} \quad k > m > l > p. \end{aligned} \tag{3}$$

The definitions of $U'_q(\mathrm{so}_n)$ by means of relations (1) and by means of relations (3) are equivalent [13]. If $q \to 1$ ('classical' limit), then both sets of relations (3), corresponding to indices '+' and '−', reduce to those of so_n.

3. Tensoring of representations of algebra $U'_q(\mathrm{so}_n)$ by the vector representation

Let T be a representation of $U'_q(\mathrm{so}_n)$ on the linear space $\mathcal{V}$ with the basis $\{v_\alpha\}$ and $\mathcal{V}_{\mathbf{1}}$ be the n-dimensional linear space with the basis $\{v_k\}$, $k = 1, 2, \ldots, n$, and $\mathcal{V}^{\otimes} \equiv \mathcal{V}_{\mathbf{1}} \otimes \mathcal{V}$.

Proposition 1. *The map $T^{\otimes}$ from $U'_q(\mathrm{so}_n)$ to* End $\mathcal{V}^{\otimes}$ *given by the formulas*

$$T^{\otimes}(I_{j,j-1})\,(v_{j-1} \otimes v_\alpha) = q\ v_{j-1} \otimes T(I_{j,j-1}) v_\alpha - q^{1/2}\ v_j \otimes v_\alpha \tag{4}$$

$$T^{\otimes}(I_{j,j-1})\,(v_j \otimes v_\alpha) = q^{-1}\ v_j \otimes T(I_{j,j-1}) v_\alpha + q^{-1/2}\ v_{j-1} \otimes v_\alpha \tag{5}$$

$$T^{\otimes}(I_{j,j-1})\,(v_k \otimes v_\alpha) = v_k \otimes T(I_{j,j-1}) v_\alpha \qquad j \neq k \quad j-1 \neq k \tag{6}$$

defines a representation of $U'_q(\mathrm{so}_n)$ on the space $\mathcal{V}^{\otimes}$.

Proof. This proposition can be proved by straightforward verification. □

In the case when T is the trivial representation of $U'_q(\mathrm{so}_n)$ given by formulas $T(a) = 0$, $a \in U'_q(\mathrm{so}_n)$, $a \neq 1$, proposition 1 gives us a representation on the space $\mathcal{V}_{\mathbf{1}} \sim \mathcal{V}^{\otimes}$. We denote this representation by $T_{\mathbf{1}}$.

$$T_{\mathbf{1}}(I_{j,j-1})\, v_k = -q^{1/2}\delta_{k,j-1}v_j + q^{-1/2}\delta_{k,j}v_{j-1}. \tag{7}$$

The representations $T_{\mathbf{1}}$ and $T_{\boldsymbol{m}_n}$, $\boldsymbol{m}_n = (1, 0, \ldots, 0)$ (see next section), are equivalent.

In the limit $q \to 1$, proposition 1 defines the representation which is the tensor product of the vector and some arbitrary representation of the Lie algebra so_n. On the basis of these two arguments, we shall also use the notation $T^{\otimes} \equiv T_{\mathbf{1}} \otimes T$.

4. Irreducible representations of $U'_q(\mathrm{so}_n)$

In this section we describe (in the framework of a q-analogue of the GT formalism) the irreducible finite-dimensional representation of the algebra $U'_q(\mathrm{so}_n)$, which are q-deformations of the finite-dimensional irreducible representations of the Lie algebra so_n. They are given by sets $\boldsymbol{m}_n$ consisting of $\lfloor n/2 \rfloor$ numbers $m_{1,n}, m_{2,n}, \ldots, m_{\lfloor n/2\rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$) which are all integral or all half-integral and satisfy the dominance conditions

$$\begin{aligned} &m_{1,2p+1} \geqslant m_{2,2p+1} \geqslant \cdots \geqslant m_{p,2p+1} \geqslant 0 \\ &m_{1,2p} \geqslant m_{2,2p} \geqslant \cdots \geqslant m_{p-1,2p} \geqslant |m_{p,2p}| \end{aligned} \tag{8}$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\boldsymbol{m}_n}$. For a basis in a representation space we take the q-analogue of the GT basis which is obtained by successive reduction of the representation $T_{\boldsymbol{m}_n}$ to the subalgebra $U'_q(\mathrm{so}_{n-1})$, $U'_q(\mathrm{so}_{n-2}), \ldots, U'_q(\mathrm{so}_3)$, $U'_q(\mathrm{so}_2) \equiv U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by the GT tableaux

$$\{\xi_n\} \equiv \{\boldsymbol{m}_n, \xi_{n-1}\} \equiv \{\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\} \equiv \cdots \equiv \{\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \ldots, \boldsymbol{m}_2\} \tag{9}$$

where the components of $\boldsymbol{m}_k$ and $\boldsymbol{m}_{k-1}$ satisfy the 'betweenness' conditions

$$\begin{aligned} &m_{1,2p+1} \geqslant m_{1,2p} \geqslant m_{2,2p+1} \geqslant m_{2,2p} \geqslant \cdots \geqslant m_{p,2p+1} \geqslant m_{p,2p} \geqslant -m_{p,2p+1} \\ &m_{1,2p} \geqslant m_{1,2p-1} \geqslant m_{2,2p} \geqslant m_{2,2p-1} \geqslant \cdots \geqslant m_{p-1,2p-1} \geqslant |m_{p,2p}|. \end{aligned} \tag{10}$$

The basis element defined by tableaux $\{\xi_n\}$ is denoted as $|\xi_n\rangle$. We suppose that the representation space is a Hilbert space and vectors $|\xi_n\rangle$ are orthonormal. It is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1 \qquad l_{j,2p} = m_{j,2p} + p - j \tag{11}$$

for the numbers $m_{i,k}$. The operator $T_{\boldsymbol{m}_n}(I_{2p+1,2p})$ of the representation $T_{\boldsymbol{m}_n}$ of $U'_q(\mathrm{so}_n)$ acts upon GT basis elements, labelled by (9), by the formula

$$T_{\boldsymbol{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} A^j_{2p}(\xi_n)|(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p} A^j_{2p}((\xi_n)^{-j}_{2p})|(\xi_n)^{-j}_{2p}\rangle \tag{12}$$

and the operator $T_{\boldsymbol{m}_n}(I_{2p,2p-1})$ of the representation $T_{\boldsymbol{m}_n}$ acts as

$$\begin{aligned} T_{\boldsymbol{m}_n}(I_{2p,2p-1})|\xi_n\rangle &= \sum_{j=1}^{p-1} B^j_{2p-1}(\xi_n)|(\xi_n)^{+j}_{2p-1}\rangle \\ &\quad - \sum_{j=1}^{p-1} B^j_{2p-1}((\xi_n)^{-j}_{2p-1})|(\xi_n)^{-j}_{2p-1}\rangle + \mathrm{i}\, C_{2p-1}(\xi_n)|\xi_n\rangle \\ T_{\boldsymbol{m}_n}(I_{21})|\xi_n\rangle &= \mathrm{i}\,[l_{12}]|\xi_n\rangle. \end{aligned} \tag{13}$$

In these formulas, $(\xi_n)_k^{\pm j}$ means the tableau (9) in which jth component $m_{j,k}$ in $\boldsymbol{m}_k$ is replaced by $m_{j,k} \pm 1$. The coefficients A^j_{2p}, B^j_{2p-1}, C_{2p-1} in (12) and (13) are given by the expressions

$$A^j_{2p}(\xi_n) = \left(\frac{[l_{j,2p}][l_{j,2p}+1]}{[2l_{j,2p}][2l_{j,2p}+2]} \frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}]} \right.$$
$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]} \right)^{\frac{1}{2}} \tag{14}$$

and

$$B^j_{2p-1}(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]}{[2l_{j,2p-1}+1][2l_{j,2p-1}-1]\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}]} \right.$$
$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-2}+l_{j,2p-1}][l_{i,2p-2}-l_{j,2p-1}]}{[l_{j,2p-1}]^2\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]} \right)^{\frac{1}{2}} \tag{15}$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p}]\prod_{i=1}^{p-1}[l_{i,2p-2}]}{\prod_{i=1}^{p-1}[l_{i,2p-1}][l_{i,2p-1}-1]} \tag{16}$$

where numbers in square brackets mean q-numbers defined by

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}}. \tag{17}$$

It is seen from (16) that C_{2p-1} in (13) identically vanishes if $m_{p,2p} \equiv l_{p,2p} = 0$.

A proof of the fact that formulas (12)–(16) indeed determine a representation of $U'_q(\mathrm{so}_n)$ is given in [9]. These representations are called representations of the classical type, since under the limit $q \to 1$ the operators $T_{\boldsymbol{m}_n}(I_{j,j-1})$ turn into the corresponding operators for irreducible finite-dimensional representations of the Lie algebra so_n with highest weights $\boldsymbol{m}_n$. Note that the algebra $U'_q(\mathrm{so}_n)$ also has irreducible finite-dimensional representations T of non-classical type (see [10]), i.e. such that the operators $T(I_{j,j-1})$ have no classical limit $q \to 1$. In this paper, we shall consider only the classical type representations.

5. Decomposition of representations $T_{\mathbf{1}} \otimes T_{\boldsymbol{m}_3}$ of the algebra $U'_q(\mathrm{so}_3)$

In this, and the following sections, we consider the decomposition of representations $T^{\otimes} \equiv T_{\mathbf{1}} \otimes T_{\boldsymbol{m}_n}$ into irreducible constituents of the algebra $U'_q(\mathrm{so}_n)$. In this section, we restrict ourselves to the case $n = 2, 3$.

First, we consider the case of the algebra $U'_q(\mathrm{so}_2) \equiv U(\mathrm{so}_2)$. This algebra has representations T_m, $m \equiv m_{12}$, $m \in \frac{1}{2}\boldsymbol{Z}$, acting on one-dimensional spaces with basis vectors $|m\rangle$ and $T_m(I_{21})|m\rangle = \mathrm{i}[m]|m\rangle$. Then

$$T^{\otimes}(I_{21})(v_1 \otimes |m\rangle) = \mathrm{i}q[m]v_1 \otimes |m\rangle - q^{1/2}v_2 \otimes |m\rangle$$
$$T^{\otimes}(I_{21})(v_2 \otimes |m\rangle) = \mathrm{i}q^{-1}[m]v_2 \otimes |m\rangle + q^{-1/2}v_1 \otimes |m\rangle.$$

This representation is reducible. We introduce the vectors

$$v_{\pm}^{(m)} = \mp\mathrm{i}q^{-1/2\pm m}v_1 + v_2. \tag{18}$$

Then the vectors $|m \pm 1\rangle^{\otimes} := v_{\pm}^{(m)} \otimes |m\rangle$ are eigenvectors of $T^{\otimes}(I_{21})$: $T^{\otimes}(I_{21})|m \pm 1\rangle^{\otimes} = \mathrm{i}[m \pm 1]|m \pm 1\rangle^{\otimes}$. This fact can be easily verified by direct calculation using the definition of q-numbers. Thus, we have decomposition $T^{\otimes} \equiv T_{\mathbf{1}} \otimes T_m = T_{m+1} \oplus T_{m-1}$.

Now, we consider the case of the algebra $U'_q(\mathrm{so}_3)$. This algebra has representations T_l, $\boldsymbol{m}_3 \equiv (m_{13}) \equiv (l)$, $l \in \{0, 1/2, 1, 3/2, \ldots\}$, acting on the spaces $\mathcal{V}_l$ with the basis vectors $|l, m\rangle$, $(m \equiv m_{12})$, $m = -l, -l+1, \ldots, l$:

$$T_l(I_{21})|l, m\rangle = \mathrm{i}[m]|l, m\rangle \qquad T_l(I_{32})|l, m\rangle = A_{l,m}|l, m+1\rangle - A_{l,m-1}|l, m-1\rangle$$

where $A_{l,m} = d_m([l-m][l+m+1])^{1/2}$, $d_m = \big([m][m+1]/([2m][2m+2])\big)^{1/2}$. Let us consider the vectors

$$|l', m\rangle^{\otimes} := \alpha^{(l')}_{l,m} v_+^{(m-1)} \otimes |l, m-1\rangle + \beta^{(l')}_{l,m} v_3 \otimes |l, m\rangle + \gamma^{(l')}_{l,m} v_-^{(m+1)} \otimes |l, m+1\rangle \tag{19}$$

where $m = -l', -l'+1, \ldots, l'$, and

$$\begin{aligned} l' &= l+1, l, l-1 && \text{if} \quad l \geqslant 1 \\ l' &= 3/2, 1/2 && \text{if} \quad l = 1/2 \\ l' &= 1 && \text{if} \quad l = 0. \end{aligned}$$

The vectors $v_\pm^{(m)}$ in (19) are defined in (18) and

$$\begin{aligned}
\alpha^{(l+1)}_{l,m} &= q^{l-m+1/2} d_{m-1}([l+m][l+m+1])^{1/2} \\
\beta^{(l+1)}_{l,m} &= ([l-m+1][l+m+1])^{1/2} \\
\gamma^{(l+1)}_{l,m} &= -q^{l+m+1/2} d_m([l-m][l-m+1])^{1/2} \\
\alpha^{(l)}_{l,m} &= -q^{-m-1/2} d_{m-1}([l+m][l-m+1])^{1/2} \\
\beta^{(l)}_{l,m} &= [m] \\
\gamma^{(l)}_{l,m} &= -q^{m-1/2} d_m([l-m][l+m+1])^{1/2} \\
\alpha^{(l-1)}_{l,m} &= -q^{-l-m-1/2} d_{m-1}([l-m][l-m+1])^{1/2} \\
\beta^{(l-1)}_{l,m} &= ([l-m][l+m])^{1/2} \\
\gamma^{(l-1)}_{l,m} &= q^{-l+m-1/2} d_m([l+m][l+m+1])^{1/2}.
\end{aligned}$$

From the case of $U'_q(\mathrm{so}_2)$, it is easy to see that $T^{\otimes}(I_{21})|l', m\rangle^{\otimes} = \mathrm{i}[m]|l', m\rangle^{\otimes}$. One can show by direct calculation that $T^{\otimes}(I_{32})|l', m\rangle^{\otimes} = A_{l',m}|l', m+1\rangle^{\otimes} - A_{l',m-1}|l', m-1\rangle^{\otimes}$. It means that the vectors $|l', m\rangle^{\otimes}$ at fixed l' span a subspace in $\mathcal{V}^{\otimes}$, which is invariant and irreducible under the action of $T^{\otimes}(a)$, $a \in U'_q(\mathrm{so}_3)$. The corresponding subrepresentation is equivalent to $T_{l'}$. Comparing the dimensions of $T_{l'}$ with the dimension of $T^{\otimes}$, we conclude that $T^{\otimes} = T_{l+1} \oplus T_l \oplus T_{l-1}$, if $l \geqslant 1$; $T^{\otimes} = T_{3/2} \oplus T_{1/2}$, if $l = 1/2$; $T^{\otimes} = T_1$, if $l = 0$. Let us recall that $T_l \equiv T_{\boldsymbol{m}_3}$, $m_{13} \equiv l$. The numbers $\alpha^{(l')}_{l,m}$, $\beta^{(l')}_{l,m}$ and $\gamma^{(l')}_{l,m}$ are CGCs of these decompositions.

6. Decomposition of $T_1 \otimes T_{\boldsymbol{m}_n}$ of the algebra $U'_q(\mathrm{so}_n)$, $n \geqslant 4$

In this section, we consider the decomposition of the representations $T^{\otimes} \equiv T_{\mathbf{1}} \otimes T_{\boldsymbol{m}_n}$ of the algebra $U'_q(\mathrm{so}_n)$, $n \geqslant 4$, into irreducible constituents. We shall show that this decomposition has the form

$$T^{\otimes} = \bigoplus_{\boldsymbol{m}'_n \in \mathcal{S}(\boldsymbol{m}_n)} T_{\boldsymbol{m}'_n} \tag{20}$$

where

$$\mathcal{S}(\boldsymbol{m}_{2p+1}) = \bigcup_{j=1}^{p} \{\boldsymbol{m}^{+j}_{2p+1}\} \cup \bigcup_{j=1}^{p} \{\boldsymbol{m}^{-j}_{2p+1}\} \cup \{\boldsymbol{m}_{2p+1}\} \tag{21}$$

$$\mathcal{S}(\boldsymbol{m}_{2p}) = \bigcup_{j=1}^{p} \{\boldsymbol{m}^{+j}_{2p}\} \cup \bigcup_{j=1}^{p} \{\boldsymbol{m}^{-j}_{2p}\}. \tag{22}$$

By $\boldsymbol{m}_n^{\pm j}$ we mean here the set $\boldsymbol{m}_n$ with $m_{j,n}$ replaced by $m_{j,n} \pm 1$, respectively. If some $\boldsymbol{m}_n^{\pm j}$ is not dominant (8), then the corresponding $\boldsymbol{m}_n^{\pm j}$ must be omitted. If $m_{p,2p+1} = 0$ then $\boldsymbol{m}_{2p+1}$ in the right-hand side of (21) must also be omitted. For decomposition (20) of the representation $T^{\otimes}$, there is a corresponding decomposition of carrier space:

$$\mathcal{V}^{\otimes} \equiv \mathcal{V}_{\mathbf{1}} \otimes \mathcal{V}_{\boldsymbol{m}_n} = \bigoplus_{\boldsymbol{m}'_n \in \mathcal{S}(\boldsymbol{m}_n)} \mathcal{V}_{\boldsymbol{m}'_n}. \tag{23}$$

In order to give this decomposition in an explicit form, we change the basis $\{v_k \otimes |\xi_n\rangle\}$, $k = 1, 2, \ldots, n$, in $\mathcal{V}^{\otimes}$ to $\{v_k \otimes |\xi_n\rangle\}$, $k = +, -, 3, \ldots, n$, by replacing (for every fixed $\{\xi_n\} = \{\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \ldots, \boldsymbol{m}_3, \boldsymbol{m}_2\}$) two basis vectors $v_1 \otimes |\xi_n\rangle$ and $v_2 \otimes |\xi_n\rangle$ by $v_+^{(m_{12})} \otimes |\xi_n\rangle$ and $v_-^{(m_{12})} \otimes |\xi_n\rangle$ (see (18)). From now on, we shall omit the index (m_{12}) in the notation of the basis vectors $v_{\pm}^{(m_{12})} \otimes |\xi_n\rangle$, supposing that it is equal to the m_{12}-component of the corresponding GT tableaux $\{\xi_n\}$.

We introduce the vectors (where $\{\xi'_n\} = \{\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \ldots, \boldsymbol{m}'_3, \boldsymbol{m}'_2\}$)

$$|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes} := \sum_k \sum_{|\boldsymbol{m}_n, \xi'_{n-1}\rangle \in \mathcal{V}_{\boldsymbol{m}_n}} \big(k, (\boldsymbol{m}_n, \xi'_{n-1}) | (\boldsymbol{m}'_n, \xi_{n-1})\big)\, v_k \otimes |\boldsymbol{m}_n, \xi'_{n-1}\rangle \tag{24}$$

in the space $\mathcal{V}^{\otimes}$, where k runs over the set $+, -, 3, \ldots, n$, and coefficients $\big(k, (\boldsymbol{m}_n, \xi'_{n-1}) | (\boldsymbol{m}'_n, \xi_{n-1})\big)$ are CGCs. We first define these CGCs in an explicit form and then, in theorem 1, prove that the formulas for the action of the operators $T^{\otimes}(I_{k+1,k})$, $k = 1, 2, \ldots, n-1$, on the vectors $|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}$ coincide with the corresponding formulas for the action of the operators $T_{\boldsymbol{m}'_n}(I_{k+1,k})$ on the GT basis vectors $|\boldsymbol{m}'_n, \xi_{n-1}\rangle$ (see (12), (13)). It will mean that the defined coefficients are really CGCs.

We put $\big(k, (\boldsymbol{m}_n, \xi'_{n-1}) | (\boldsymbol{m}'_n, \xi_{n-1})\big) = 0$ if one of the conditions

(1) $\boldsymbol{m}'_n \notin \mathcal{S}(\boldsymbol{m}_n)$		
(2) $\boldsymbol{m}_s \notin \mathcal{S}(\boldsymbol{m}'_s)$	$s = n-1, \ldots, k$	$k \geqslant 3$
(3) $\boldsymbol{m}_s \notin \mathcal{S}(\boldsymbol{m}'_s)$	$s = n-1, \ldots, 3$	$k = +, -$
(4) $\xi'_{k-1} \neq \xi_{k-1}$	$k = 3, 4, \ldots, n$	
(5) $m_{12} \neq m'_{12} + 1$	$k = +$	
(6) $m_{12} \neq m'_{12} - 1$	$k = -$	

is fulfilled. The non-zero CGC for $k = n$ are:

$$\begin{aligned}
\big(2p+1, (\boldsymbol{m}_{2p+1}, \xi_{2p}) | (\boldsymbol{m}_{2p+1}^{+j}, \xi_{2p})\big) &= \left(\prod_{r=1}^{p} [l_{j,2p+1} + l_{r,2p}][l_{j,2p+1} - l_{r,2p}]\right)^{\frac{1}{2}} \\
\big(2p+1, (\boldsymbol{m}_{2p+1}, \xi_{2p}) | (\boldsymbol{m}_{2p+1}, \xi_{2p})\big) &= \prod_{r=1}^{p} [l_{r,2p}] \\
\big(2p+1, (\boldsymbol{m}_{2p+1}, \xi_{2p}) | (\boldsymbol{m}_{2p+1}^{-j}, \xi_{2p})\big) &= \left(\prod_{r=1}^{p} [l_{j,2p+1} + l_{r,2p} - 1][l_{j,2p+1} - l_{r,2p} - 1]\right)^{\frac{1}{2}}
\end{aligned} \tag{25}$$

$$\begin{aligned}
\big(2p, (\boldsymbol{m}_{2p}, \xi_{2p-1}) | (\boldsymbol{m}_{2p}^{+j}, \xi_{2p-1})\big) &= \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1}][l_{j,2p} - l_{r,2p-1} + 1]\right)^{\frac{1}{2}} \\
\big(2p, (\boldsymbol{m}_{2p}, \xi_{2p-1}) | (\boldsymbol{m}_{2p}^{-j}, \xi_{2p-1})\big) &= \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1} - 1][l_{j,2p} - l_{r,2p-1}]\right)^{\frac{1}{2}}.
\end{aligned} \tag{26}$$

(They are defined up to normalization, i.e. multiplication of these CGCs by some constants will not spoil the following results.)

All the other CGCs can be represented by the following formula:

$$\big(k, \xi_n|\xi'_n\big) = q^{k-n} \frac{\langle \boldsymbol{m}_{n+1}, \xi_n | T_{\boldsymbol{m}_{n+1}}(I^-_{n+1,k}) | \boldsymbol{m}_{n+1}, \xi'_n\rangle}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \xi_{n-1} | T_{\boldsymbol{m}_{n+1}}(I_{n+1,n}) | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \xi_{n-1}\rangle} \times \big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big) \tag{27}$$

where the generators $I^-_{n+1,k}$ are defined in (2). If $k = +$ or $k = -$ in the left-hand side of (27), one must put $k = 2$ in the right-hand side. The set $\boldsymbol{m}_{n+1}$ must be chosen to give a non-zero denominator in the right-hand side of (27). Note that if $(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})) \neq 0$, one can always make such a choice; moreover, the resulting CGC will not depend on this particular choice. In the case $n = 3$ we reobtain the CGCs for the algebra $U'_q(\mathrm{so}_3)$ (see the previous section).

Before we prove the main theorem, we describe the CGCs *factorization* property. Let us consider the CGC $(n-1, (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}))$. It depends on $\boldsymbol{m}_n$, $\boldsymbol{m}'_n$, $\boldsymbol{m}_{n-1}$, $\boldsymbol{m}'_{n-1}$ and $\boldsymbol{m}_{n-2}$. From (12), (13) and (25)–(27), it follows that all the dependence on the numbers $\boldsymbol{m}_{n-2}$ in this coefficient appears to be due to the matrix element of operator $T_{\boldsymbol{m}_{n+1}}(I^-_{n+1,n-1})$, where $I^-_{n+1,n-1} = q^{-1/2} I_{n,n-1} I_{n+1,n} - q^{1/2} I_{n+1,n} I_{n,n-1}$. This dependence, which in fact arises from some factors in matrix elements of operator $T_{\boldsymbol{m}_{n+1}}(I_{n,n-1})$, has the form of the CGC $(n-1, (\boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_{n-1}, \xi_{n-2}))$ for the algebra $U'_q(\mathrm{so}_{n-1})$. This fact follows from the explicit expressions of matrix elements (14)–(16) and from the formulas (25), (26) with n replaced by $n-1$. Thus, we have the decomposition

$$\big(n-1, \xi_n|\xi'_n\big) = \left(\begin{array}{cc|c} 1 & \boldsymbol{m}_n & \boldsymbol{m}'_n \\ 1 & \boldsymbol{m}_{n-1} & \boldsymbol{m}'_{n-1} \end{array}\right) \big(n-1, \xi_{n-1}|\xi'_{n-1}\big). \tag{28}$$

The expression $\left(\begin{array}{cc|c} 1 & \boldsymbol{m}_n & \boldsymbol{m}'_n \\ 1 & \boldsymbol{m}_{n-1} & \boldsymbol{m}'_{n-1} \end{array}\right)$, uniquely defined by this decomposition, does not depend either on ξ_{n-2} or on ξ'_{n-2}.

Proposition 2 (CGCs factorization property). *We have the following decomposition*

$$\big(k, \xi_n|\xi'_n\big) = \left(\begin{array}{cc|c} 1 & \boldsymbol{m}_n & \boldsymbol{m}'_n \\ 1 & \boldsymbol{m}_{n-1} & \boldsymbol{m}'_{n-1} \end{array}\right) \big(k, \xi_{n-1}|\xi'_{n-1}\big) \qquad k < n. \tag{29}$$

Proof. Let us consider the product of $\left(\begin{array}{cc|c} 1 & \boldsymbol{m}_n & \boldsymbol{m}'_n \\ 1 & \boldsymbol{m}_{n-1} & \boldsymbol{m}'_{n-1} \end{array}\right)$ with an arbitrary CGC for the algebra $U'_q(\mathrm{so}_{n-1})$ given by formula (27) with n replaced by $n-1$. (Hereafter, for convenience, by the symbol $I^-_{k,l}$ we also mean the corresponding representation operator $T_{\boldsymbol{m}_{n+1}}(I^-_{k,l})$.)

$$\begin{aligned}
&\left(\begin{array}{cc|c} 1 & \boldsymbol{m}_n & \boldsymbol{m}'_n \\ 1 & \boldsymbol{m}_{n-1} & \boldsymbol{m}'_{n-1} \end{array}\right) \big(k, \xi_{n-1}|\xi'_{n-1}\big) \\
&\quad = q^{-1} \frac{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2} | I^-_{n+1,n-1} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \xi_{n-1} | I_{n+1,n} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \xi_{n-1}\rangle} \\
&\quad \times \frac{\big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big)}{\big(n-1, (\boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_{n-1}, \xi_{n-2})\big)} \big(n-1, (\boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_{n-1}, \xi_{n-2})\big) \\
&\quad \times q^{k-(n-1)} \frac{\langle \boldsymbol{m}_{n+1}, \tilde{\boldsymbol{m}}_n, \xi_{n-1} | I^-_{n,k} | \boldsymbol{m}_{n+1}, \tilde{\boldsymbol{m}}_n, \xi'_{n-1}\rangle}{\langle \boldsymbol{m}_{n+1}, \tilde{\boldsymbol{m}}_n, \boldsymbol{m}_{n-1}, \xi_{n-2} | I_{n,n-1} | \boldsymbol{m}_{n+1}, \tilde{\boldsymbol{m}}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle}
\end{aligned} \tag{30}$$

where all the $\tilde{\boldsymbol{m}}_n$ are equal to $\boldsymbol{m}'_n$ or $\boldsymbol{m}_n$ (as noted earlier, both these variants lead to the same CGC $(k, \xi_{n-1}|\xi'_{n-1})$). Now we decompose the obtained expression into two summands

$$\begin{aligned}
&\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2} | I^-_{n+1,n-1} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle \\
&\quad = q^{-1/2} \langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2} | I^-_{n,n-1} I^-_{n+1,n} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle \\
&\quad - q^{1/2} \langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2} | I^-_{n+1,n} I^-_{n,n-1} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle.
\end{aligned}$$

Fixing $\tilde{\boldsymbol{m}}_n = \boldsymbol{m}_n$ for the first summand and $\tilde{\boldsymbol{m}}_n = \boldsymbol{m}'_n$ for the second summand, we have for (30) the following expression

$$\frac{q^{k-n}\langle \boldsymbol{m}_{n+1}, \xi_n | I^-_{n+1,k} | \boldsymbol{m}_{n+1}, \xi'_n\rangle}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \xi_{n-1} | I_{n+1,n} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \xi_{n-1}\rangle}\big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big) = \big(k, \xi_n|\xi'_n\big).$$

Thus, the proposition is proved. □

Iterating proposition 2, we can present an arbitrary CGC for the algebra $U'_q(\mathrm{so}_n)$ as a product of expressions of type $\left(\begin{smallmatrix} 1 & \boldsymbol{m}_k \\ 1 & \boldsymbol{m}_{k-1}\end{smallmatrix}\middle|\begin{smallmatrix} \boldsymbol{m}'_k \\ \boldsymbol{m}'_{k-1}\end{smallmatrix}\right)$ and (25), (26).

Theorem 1. *The operators $T^{\otimes}(I_{k+1,k})$, $k = 1, 2, \ldots, n-1$, act on the set of the vectors $\{|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}\}$ defined by (24) with CGCs defined by (25)–(27), in correspondence with (12), (13). We have the decomposition (20).*

Proof. We shall prove the theorem by induction. As claimed in the previous section, this theorem is valid for the case of the algebra $U'_q(\mathrm{so}_3)$. We assume that the theorem is valid for the case of the algebra $U'_q(\mathrm{so}_{n-1})$, and then prove this theorem for the case of the algebra $U'_q(\mathrm{so}_n)$.

First, let us show that the action of the operators $T^{\otimes}(I_{k+1,k})$, $k = 1, 2, \ldots, n-2$, on the vectors $\{|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}\}$ is correct, i.e. this action exactly corresponds to the action of $T_{\boldsymbol{m}_{n-1}}(I_{k+1,k})$ on the basis vectors $\{|\boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\}$. Using (24) and proposition 2, we have

$$|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes} = \big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big)\big\{v_n \otimes |\boldsymbol{m}_n, \xi_{n-1}\rangle\big\} + \sum_{\boldsymbol{m}'_{n-1}} \left(\begin{matrix} 1 & \boldsymbol{m}_n \\ 1 & \boldsymbol{m}'_{n-1}\end{matrix}\middle|\begin{matrix} \boldsymbol{m}'_n \\ \boldsymbol{m}_{n-1}\end{matrix}\right)$$
$$\times\left\{\sum_{k=n-1,\ldots,3,+,-}\ \sum_{\xi'_{n-2}}\big(k, (\boldsymbol{m}'_{n-1}, \xi'_{n-2})|(\boldsymbol{m}_{n-1}, \xi_{n-2})\big) v_k \otimes |\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi'_{n-2}\rangle\right\}.$$

The vectors in the braces transform under the action of operators $T^{\otimes}(I_{k+1,k})$, $k = 1, 2, \ldots, n-2$, as vectors $\{|\boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\}$ under the action of operators $T_{\boldsymbol{m}_{n-1}}(I_{k+1,k})$. For the vector in the first braces, this fact follows from formulas (6). For the vector in the second braces, this fact follows from the coincidence of this vector with the right-hand side of (24) with n replaced by $n-1$ and from the induction assumption. Since coefficients at these braces do not depend on ξ_{n-2}, we obtain that the action of operators $T^{\otimes}(I_{k+1,k})$, $k = 1, 2, \ldots, n-2$, on vectors $\{|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}\}$ is as required.

Now, let us show that the action of the operator $T^{\otimes}(I_{n,n-1})$ on the vectors $\{|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}\}$ is also correct. For this end, we write down the required action (corresponding to the formulas (12), (13)) of the operator $T^{\otimes}(I_{n,n-1})$ on the vector $|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}$:

$$T^{\otimes}(I_{n,n-1})|\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle^{\otimes}$$
$$= \sum_{\boldsymbol{m}'_{n-1}\in\mathcal{S}(\boldsymbol{m}_{n-1})} \langle \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I_{n,n-1}|\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle|\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle^{\otimes}$$

where the map $\mathcal{S}$ is defined in (21), (22). Then, we rewrite the right-hand side of this relation using (24) for $|\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle^{\otimes}$, and compare this result with the result obtained by direct action of $T^{\otimes}(I_{n,n-1})$ on the right-hand side of (24) for $|\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle^{\otimes}$ and using formulas (4)–(6). The results must be the same. To prove this we need to show that the coefficients at the different basis vectors of these results are equal. We shall illustrate three cases.

- Basis vector $v_n \otimes |\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle$, $\boldsymbol{m}'_{n-1} \in \mathcal{S}(\boldsymbol{m}_{n-1})$.
  Equating the coefficients at the vector $v_n \otimes |\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle$ we obtain the relation

$$\begin{aligned}
&\langle \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I_{n,n-1}|\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\big(n, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})\big)\\
&\quad = q^{-1}\langle \boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I_{n,n-1}|\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\\
&\qquad \times\big(n, (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big)\\
&\qquad - q^{1/2}\big(n-1, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big).
\end{aligned}$$

  Using the identity

$$\begin{aligned}
&\big(n, (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big) = \big(n, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})\big)\\
&\qquad \times \frac{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2}|I_{n+1,n}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I_{n+1,n}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle}
\end{aligned}$$

  which follows from (14)–(16) and (25), (26), we rewrite the previous relation in the following equivalent form

$$\begin{aligned}
&\big(n-1, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big)\\
&\quad = q^{-1}\frac{\big(n, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})\big)}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I_{n+1,n}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle}\\
&\qquad \times\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I^-_{n+1,n-1}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle.
\end{aligned}$$

  The obtained relation directly follows from definition (27) of the CGC for $k = n-1$, and therefore is correct.
- Basis vector $v_{n-1} \otimes |\boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \xi_{n-2}\rangle$, $\boldsymbol{m}''_{n-1} \in \mathcal{S}(\boldsymbol{m}'_{n-1})$, $\boldsymbol{m}'_{n-1} \in \mathcal{S}(\boldsymbol{m}_{n-1})$.
  Equating the coefficients at the vector $v_{n-1} \otimes |\boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \xi_{n-2}\rangle$ we obtain the relation

$$\begin{aligned}
&\sum_{\boldsymbol{m}'_{n-1}} \langle \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}|I_{n,n-1}|\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\\
&\qquad \times\big(n-1, (\boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})\big)\\
&\quad = \sum_{\boldsymbol{m}'_{n-1}} q\,\big(n-1, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big)\\
&\qquad \times\langle \boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \xi_{n-2}|I_{n,n-1}|\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \xi_{n-2}\rangle\\
&\qquad + \delta_{\boldsymbol{m}_{n-1},\boldsymbol{m}''_{n-1}} q^{-1/2}\big(n, (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big)
\end{aligned}$$

  which acquires the equivalent form:

$$\begin{aligned}
&q^{-1}\,\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \xi_{n-2}|I^-_{n+1,n-1}I_{n,n-1}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\\
&\quad = \langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \xi_{n-2}|I_{n,n-1}I^-_{n+1,n-1}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle\\
&\qquad + \delta_{\boldsymbol{m}_{n-1},\boldsymbol{m}''_{n-1}} q^{-1/2}\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2}|I_{n+1,n}\\
&\qquad \times|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle
\end{aligned}\tag{31}$$

  if one uses the identity

$$\begin{aligned}
&\big(n-1, (\boldsymbol{m}_n, \tilde{\boldsymbol{m}}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \breve{\boldsymbol{m}}_{n-1}, \xi_{n-2})\big)\\
&\quad = q^{-1}\frac{\big(n, (\boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2})|(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2})\big)}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}_{n-1}, \xi_{n-2}|I_{n+1,n}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle}\\
&\qquad \times\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \tilde{\boldsymbol{m}}_{n-1}, \xi_{n-2}|I^-_{n+1,n-1}|\boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \breve{\boldsymbol{m}}_{n-1}, \xi_{n-2}\rangle
\end{aligned}$$

  with $\tilde{\boldsymbol{m}}_{n-1} = \boldsymbol{m}''_{n-1}$, $\breve{\boldsymbol{m}}_{n-1} = \boldsymbol{m}'_{n-1}$ and with $\tilde{\boldsymbol{m}}_{n-1} = \boldsymbol{m}'_{n-1}$, $\breve{\boldsymbol{m}}_{n-1} = \boldsymbol{m}_{n-1}$. The equality (31) directly follows from the identity $[I^-_{n+1,n-1}, I_{n,n-1}]_{q^{-1}} = I_{n+1,n}$ (see (3)).

- Basis vector $v_k \otimes |\boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1}\rangle$, $\boldsymbol{m}''_{n-1} \in \mathcal{S}(\boldsymbol{m}'_{n-1})$, $\boldsymbol{m}'_j \in \mathcal{S}(\boldsymbol{m}_j)$, $j = k, k+1, \ldots, n-1$, $k = n-2, n-3, \ldots, +, -$.
  Equating the coefficients at the vector $v_k \otimes |\boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1}\rangle$, $k < n-1$, we obtain the relation

$$\begin{aligned}
&\sum_{\boldsymbol{m}'_{n-1}} \langle \boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \xi_{n-2} | I_{n,n-1} | \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \xi_{n-2}\rangle \\
&\qquad \times \big(k, (\boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1}) \\
&\qquad \times |(\boldsymbol{m}'_n, \boldsymbol{m}'_{n-1}, \boldsymbol{m}_{n-2}, \ldots, \boldsymbol{m}_k, \xi_{k-1})\big) \\
&\qquad = \sum_{\boldsymbol{m}'_{n-1}} \big(k, (\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1}) \\
&\qquad \times |(\boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \boldsymbol{m}_{n-2}, \ldots, \boldsymbol{m}_k, \xi_{k-1})\big) \\
&\qquad \times \langle \boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1} | I_{n,n-1} \\
&\qquad \times |\boldsymbol{m}_n, \boldsymbol{m}'_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1}\rangle.
\end{aligned}$$

  Using (27), it can be rewritten in the compact form:

$$\begin{aligned}
&\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1} | I^-_{n+1,k} \\
&\qquad \times I_{n,n-1} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \boldsymbol{m}_{n-2}, \ldots, \boldsymbol{m}_k, \xi_{k-1}\rangle \\
&\qquad = \langle \boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \boldsymbol{m}''_{n-1}, \boldsymbol{m}'_{n-2}, \ldots, \boldsymbol{m}'_k, \xi_{k-1} | I_{n,n-1} \\
&\qquad \times I^-_{n+1,k} | \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \boldsymbol{m}_{n-1}, \boldsymbol{m}_{n-2}, \ldots, \boldsymbol{m}_k, \xi_{k-1}\rangle
\end{aligned}$$

  (if index k at v_k is '+' or '−', one should replace $I^-_{n+1,k} \to I^-_{n+1,2}$). This equality directly follows from the identity $[I^-_{n+1,k}, I_{n,n-1}] = 0$ (see (3)).

In order to prove the decomposition (20), we need to show that the space $\mathcal{V}^{\otimes}$ is the *direct* sum of the described subspaces $\mathcal{V}_{\boldsymbol{m}'_n}$. It can be shown by comparing the corresponding dimensions, using the fact that all the representations $T_{\boldsymbol{m}'_n}$ are pairwise inequivalent. Thus, the theorem is proved. □

7. Embedding $U'_q(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$

The quantum algebra $U_q(\mathrm{sl}_n)$ is defined [1, 2] as a complex associative algebra with the generating elements e_i, f_i, k_i, k_i^{-1}, $i = 1, 2, \ldots, n-1$ and defining relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad k_i k_j = k_j k_i \qquad k_i e_j k_i^{-1} = q^{a_{ij}} e_j \qquad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j$$

$$[e_i, e_j] = [f_i, f_j] = 0 \qquad |i-j| > 1 \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$$e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0 \qquad f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0$$

where $a_{ii} = 2$, $a_{i,i\pm1} = -1$ and $a_{ij} = 0$ for $|i-j| > 1$. Here q is a complex parameter, $q \neq 0, \pm1$. It is shown in [11,12], that the elements $\tilde{I}_{i+1,i} = f_i - q^{-1} k_i e_i$, $i = 1, 2, \ldots, n-1$, satisfy relations (1) and define a homomorphism $U'_q(\mathrm{so}_n) \to U_q(\mathrm{sl}_n)$. Moreover, it is proved in [13] that this homomorphism is an embedding, i.e. we may consider $U'_q(\mathrm{so}_n)$ as a subalgebra in $U_q(\mathrm{sl}_n)$.

Let us define representation $\mathcal{T}_{\mathbf{1}}$ of $U_q(\mathrm{sl}_n)$ on n-dimensional space $\mathcal{V}_{\mathbf{1}}$ with the basis vectors v_k, $k = 1, 2, \ldots, n$, by the formulas

$$\begin{aligned}
&\mathcal{T}_{\mathbf{1}}(e_i)\, v_k = -q^{-1/2} \delta_{i+1,k} v_{k-1} \qquad \mathcal{T}_{\mathbf{1}}(f_i)\, v_k = -q^{1/2} \delta_{i,k} v_{k+1} \\
&\mathcal{T}_{\mathbf{1}}(k_i)\, v_k = q^{\delta_{i,k} - \delta_{i+1,k}} v_k.
\end{aligned} \tag{32}$$

It is easy to verify that this representation is the vector representation (i.e. representation with the highest weight $(1, 0, \ldots, 0)$). The action formulas (32) imply

$$\mathcal{T}_{\mathbf{1}}(\tilde{I}_{i+1,i})\, v_k = -q^{1/2}\delta_{i,k}v_{k+1} + q^{-1/2}\delta_{i+1,k}v_{k-1}. \tag{33}$$

This representation of $U'_q(\mathrm{so}_n)$ (see (7)) is equivalent to the classical type representation $T_{\boldsymbol{m}_n}$ with $\boldsymbol{m}_n = (1, 0, \ldots, 0)$ (the vector representation). Hence, similar to the classical case, the restriction of the vector representation of $U_q(\mathrm{sl}_n)$ onto $U'_q(\mathrm{so}_n)$ is the vector representation of $U'_q(\mathrm{so}_n)$.

The quantum algebra $U_q(\mathrm{sl}_n)$ possesses the Hopf structure. Comultiplication on generators of this algebra can be defined as

$$\Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i \qquad \Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i \qquad \Delta(k_i) = k_i \otimes k_i.$$

Therefore,

$$\Delta(\tilde{I}_{i+1,i}) = \tilde{I}_{i+1,i} \otimes 1 + k_i \otimes \tilde{I}_{i+1,i}. \tag{34}$$

Let $\mathcal{T}$ be a representation of $U_q(\mathrm{sl}_n)$ on the space $\mathcal{V}$. Then, using the comultiplication, we can define the representation $\mathcal{T}^{\otimes} := \mathcal{T}_{\mathbf{1}} \otimes \mathcal{T}$ on the space $\mathcal{V}_{\mathbf{1}} \otimes \mathcal{V}$. From (33) and (34), formulas (4)–(6) follow, where T is the representation $\mathcal{T}$ of $U_q(\mathrm{sl}_n)$ restricted to $U'_q(\mathrm{so}_n)$. Moreover, the algebra $U'_q(\mathrm{so}_n)$ is a $U_q(\mathrm{sl}_n)$-comodule algebra with the coaction $\phi(I_{i+1,i}) = \tilde{I}_{i+1,i} \otimes 1 + k_i \otimes I_{i+1,i}$. Embedding $U'_q(\mathrm{so}_n)$ into $U_q(\mathrm{sl}_n)$ we obtain (34). Proposition 1 is a straightforward sequence of this comodule algebra structure.

Theorem 2. *Every irreducible finite-dimensional representation $\mathcal{T}$ of $U_q(\mathrm{sl}_n)$ considered as a representation of $U'_q(\mathrm{so}_n)$ is a direct sum of irreducible classical type representations $T_{\boldsymbol{m}_n}$ of $U'_q(\mathrm{so}_n)$ with integral m_{jn} in $\boldsymbol{m}_n$.*

Proof. At first, we prove this theorem for the case of irreducible finite-dimensional type 1 representations of $U_q(\mathrm{sl}_n)$. Every such representation is uniquely determined (see [16, section 7.1.2]) by the highest weight of an irreducible finite-dimensional representation of the Lie algebra sl_n, i.e. by the set of *non-negative* integers $\boldsymbol{\mu}_n = (\mu_{1,n}, \mu_{2,n}, \ldots, \mu_{n-1,n})$, $\mu_{j,n} = 2(\boldsymbol{\mu}_n, \alpha_j)/(\alpha_j, \alpha_j)$, (where α_j are simple positive roots of the Lie algebra sl_n). For this set, we define $|\boldsymbol{\mu}_n| = \mu_{1,n} + 2\mu_{2,n} + \cdots + (n-1)\mu_{n-1,n}$, and prove the theorem by induction on $|\boldsymbol{\mu}_n|$. There is only one representation of $U_q(\mathrm{sl}_n)$ with $|\boldsymbol{\mu}_n| = 1$. It is the vector representation ($\boldsymbol{\mu}_n = (1, 0, \ldots, 0)$). As was claimed above, the restriction of this representation onto $U'_q(\mathrm{so}_n)$ is equivalent to the classical type representation $T_{\boldsymbol{m}_n}$ with $\boldsymbol{m}_n = (1, 0, \ldots, 0)$. Thus, the theorem is valid for representations $\mathcal{T}_{\mu_n}$ with $|\boldsymbol{\mu}_n| = 1$.

Now we assume that the theorem is valid for all the representations $\mathcal{T}_{\mu_n}$ with $|\boldsymbol{\mu}_n| \leqslant s$. Let us consider a representation $\mathcal{T}_{\mu'_n}$ with $|\boldsymbol{\mu}'_n| = s + 1$. It is well known (see [16, section 7.2.1]), that for every representation $\mathcal{T}_{\mu'_n}$ of $U_q(\mathrm{sl}_n)$ with $|\boldsymbol{\mu}'_n| \geqslant 1$, there exists a representation $\mathcal{T}_{\mu_n}$ with $|\boldsymbol{\mu}_n| = |\boldsymbol{\mu}'_n| - 1$ such that $\mathcal{T}_{\mathbf{1}} \otimes \mathcal{T}_{\mu_n}$ contains $\mathcal{T}_{\mu'_n}$ as an irreducible subrepresentation. This fact, theorem 1 and the induction assumption lead us to the fact that the restriction of representation $\mathcal{T}_{\mathbf{1}} \otimes \mathcal{T}_{\mu_n}$, (and, therefore, the representation $\mathcal{T}_{\mu'_n}$) onto $U'_q(\mathrm{so}_n)$ contains only irreducible classical type representations $T_{\boldsymbol{m}_n}$ of $U'_q(\mathrm{so}_n)$ with integral m_{jn} in $\boldsymbol{m}_n$.

Every irreducible finite-dimensional representation $\mathcal{T}$ of $U_q(\mathrm{sl}_n)$ can be presented as $\mathcal{T} = \mathcal{T}_{\mu_n} \otimes \mathcal{T}_{\tilde{\mathbf{1}}}$, where $\mathcal{T}_{\mu_n}$ is an irreducible finite-dimensional type 1 representation, and $\mathcal{T}_{\tilde{\mathbf{1}}}$ is a one-dimensional representation of $U_q(\mathrm{sl}_n)$ with matrices $\mathcal{T}_{\tilde{\mathbf{1}}}(e_i) = \mathcal{T}_{\tilde{\mathbf{1}}}(f_i) = 0$. Therefore, $\mathcal{T}_{\tilde{\mathbf{1}}}(\tilde{I}_{i+1,i}) = 0$. From formula (34), it follows that representations $\mathcal{T}$ and $\mathcal{T}_{\mu_n}$ considered as representations of $U'_q(\mathrm{so}_n)$ have equivalent decomposition into irreducible constituents. □

Conjecture. *Decomposition of irreducible representation $\mathcal{T}_{\mu_n}$ of $U_q(\mathrm{sl}_n)$ restricted to $U'_q(\mathrm{so}_n)$ contains the representations $T_{\boldsymbol{m}_n}$ of $U'_q(\mathrm{so}_n)$ with the same multiplicities as in the case of corresponding Lie algebra.*

We have proof of this conjecture for the cases of some particular representations $\mathcal{T}_{\mu_n}$ such as symmetrical, antisymmetrical (fundamental) and some other ones. For the proof of this conjecture for the case of $n = 3$, see [17].

Theorem 3. *The set of irreducible finite-dimensional classical type representations $T_{\boldsymbol{m}_n}$ of $U_q'(\mathrm{so}_n)$ with integral m_{jn} in $\boldsymbol{m}_n$ separates the elements of this algebra, i.e. for two arbitrary elements $X, Y \in U_q'(\mathrm{so}_n)$, $X \neq Y$, there exists a representation $T_{\boldsymbol{m}_n}$ such that $T_{\boldsymbol{m}_n}(X) \neq T_{\boldsymbol{m}_n}(Y)$.*

Proof. It was shown in [13] that the irreducible representations of $U_q'(\mathrm{so}_n)$ obtained from restriction of irreducible finite-dimensional representations of $U_q(\mathrm{sl}_n)$ separate the elements of the algebra $U_q'(\mathrm{so}_n)$. This fact and theorem 2 imply theorem 3. □

8. The Wigner–Eckart theorem for the vector operators

The formula (24) gives us the transformation from the basis $\{v_k \otimes |\xi_n\rangle\}$ to the basis $\{|\xi_n'\rangle^\otimes\}$ in the space $\mathcal{V}^\otimes$. Because of (23), the transformation matrix is a non-degenerate matrix with matrix element CGCs $(k, \xi_n|\xi_n')$. Denote the matrix elements of the inverse matrix by $(\xi_n'|k, \xi_n)$ (*inverse* CGCs). Let us find the expression for the vector $v_n \otimes |\boldsymbol{m}_n, \xi_{n-1}\rangle$ from (24) in terms of vectors $|\xi_n'\rangle^\otimes$. Since this vector transforms under the action of $T^\otimes(a)$, $a \in U_q'(\mathrm{so}_{n-1})$, as the vector $|\xi_{n-1}\rangle$ under the action of $T_{\boldsymbol{m}_{n-1}}(a)$ (see formula (6)), from the Schur lemma we have

$$v_n \otimes |\boldsymbol{m}_n, \xi_{n-1}\rangle = \sum_{\boldsymbol{m}_n'} \big((\boldsymbol{m}_n', \xi_{n-1})|n, (\boldsymbol{m}_n, \xi_{n-1})\big)\, |\boldsymbol{m}_n', \xi_{n-1}\rangle^\otimes \tag{35}$$

where the coefficients $\big((\boldsymbol{m}_n', \xi_{n-1})|n, (\boldsymbol{m}_n, \xi_{n-1})\big)$ depend only on $\boldsymbol{m}_n', \boldsymbol{m}_n, \boldsymbol{m}_{n-1}$. From (24), it also follows that $\boldsymbol{m}_n' \in \mathcal{S}(\boldsymbol{m}_n)$. Although these coefficients are uniquely defined by (24)–(27), we only need their explicit dependence on $\boldsymbol{m}_{n-1}$.

Let T° (respectively T) be a finite-dimensional representation of some associative algebra $\mathcal{A}$ on a space $\mathcal{V}^\circ$ with the basis $\{v_k\}$ (respectively on a space $\mathcal{V}$ with the basis $\{v_\alpha\}$). If we have a definition of representation $T^\circ \otimes T$ on $\mathcal{V}^\circ \otimes \mathcal{V}$, we can introduce the notion of tensor operator on $\mathcal{V}$.

Definition 1. *The set $\{V_k\}$ of* $\dim \mathcal{V}^\circ$ *operators on $\mathcal{V}$ is called tensor operator of $\mathcal{A}$ transforming as T° if for all $a \in U_q'(\mathrm{so}_n)$, T, v_k, v_α we have*

$$\sigma \circ (T^\circ \otimes T)(a) v_k \otimes v_\alpha = (T(a) \circ \sigma) v_k \otimes v_\alpha$$

where σ is a linear map $\mathcal{V}^\circ \otimes \mathcal{V} \to \mathcal{V}$, such that $\sigma(v_k \otimes v_\alpha) = V_k v_\alpha$.

Proposition 1 and the above definition give us a possibility to introduce the notation of the vector operator of $U_q'(\mathrm{so}_n)$ on the space $\mathcal{V}$. Substituting $T^\circ \to T_{\boldsymbol{1}}$, $T^\circ \otimes T \to T^\otimes$, we obtain the following definition.

Definition 2. *The set $\{V_k\}$, $k = 1, 2, \ldots, n$, of operators on $\mathcal{V}$, where a representation T of $U_q'(\mathrm{so}_n)$ is realized, such that*

$$[V_{j-1}, T(I_{j,j-1})]_q = V_j \qquad [T(I_{j,j-1}), V_j]_q = V_{j-1} \tag{36}$$

$$[T(I_{j,j-1}), V_k] = 0 \qquad \text{if} \quad j \neq k \quad \text{and} \quad j-1 \neq k \tag{37}$$

where $[X, Y]_q = q^{1/2}XY - q^{-1/2}YX$, is called the vector operator of the algebra $U_q'(\mathrm{so}_n)$.

It is easy to verify that the action of operators $T(I_{j,j-1})$ on the vectors $V_k\, v_\alpha$ directly corresponds to actions (4)–(6) of operators $T^{\otimes}(I_{j,j-1})$ on the vectors $v_k \otimes v_\alpha$.

Let T be a direct sum of irreducible classical type representations of $U'_q(\mathrm{so}_n)$ with arbitrary multiplicities. Choose a GT basis in $\mathcal{V}$. Let us consider an invariant subspaces $\mathcal{V}_{\boldsymbol{m}_n,s}$ where subrepresentation equivalent to $T_{\boldsymbol{m}_n}$ is realized. The number s labels the number of such subspaces if the corresponding multiplicity exceeds 1. Combine the vectors $V_k\,|(\boldsymbol{m}_n, \xi_{n-1}); s\rangle$, where $\{|(\boldsymbol{m}_n, \xi_{n-1}); s\rangle\}$ is a GT basis of $\mathcal{V}_{\boldsymbol{m}_n,s}$, with the CGC as in (24) for some fixed $\boldsymbol{m}'_n \in \mathcal{S}(\boldsymbol{m}_n)$. Two variants are possible. First, all the vectors $|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}$ are zero. Second, on the space spanned by the vectors $|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes}$, a representation of $U'_q(\mathrm{so}_n)$ equivalent to $T_{\boldsymbol{m}'_n}$ is realized. From the Schur lemma, it follows that

$$|\boldsymbol{m}'_n, \xi_{n-1}\rangle^{\otimes} = \sum_{s'} (\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)\, |\boldsymbol{m}'_n, \xi_{n-1}; s'\rangle \tag{38}$$

where $(\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)$ are some coefficients (*reduced matrix elements*) depending only on $\boldsymbol{m}'_n, s', \boldsymbol{m}_n, s$ and the vector operator $\{V_k\}$. Using the analogue of relation (35) for the vector operator and (38) we have

$$V_n|\boldsymbol{m}_n, \xi_{n-1}; s\rangle = \sum_{\boldsymbol{m}'_n, s'} \big((\boldsymbol{m}'_n, \xi_{n-1})|n, (\boldsymbol{m}_n, \xi_{n-1})\big)(\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)\, |\boldsymbol{m}'_n, \xi_{n-1}; s'\rangle. \tag{39}$$

As was claimed above, the coefficients $\big((\boldsymbol{m}'_n, \xi_{n-1})|n, (\boldsymbol{m}_n, \xi_{n-1})\big)$ may depend on $\boldsymbol{m}_{n-1}$. Since this dependence is identical for all the possible vector operators in arbitrary spaces, we choose, for a moment, $\mathcal{V}$ to be the space $\mathcal{V}_{\boldsymbol{m}_{n+1}}$ of the irreducible representation $T_{\boldsymbol{m}_{n+1}}$ of $U'_q(\mathrm{so}_{n+1})$ for some convenient $\boldsymbol{m}_{n+1}$, and $\{V_k\} \equiv \{T_{\boldsymbol{m}_{n+1}}(I^+_{n+1,k})\}$. Extracting the dependence on $\boldsymbol{m}_{n-1}$ from the matrix elements of $T_{\boldsymbol{m}_{n+1}}(I_{n+1,n})$ and comparing it with formulas (25), (26), we obtain

$$\big((\boldsymbol{m}'_n, \xi_{n-1})|n, (\boldsymbol{m}_n, \xi_{n-1})\big) = \big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big)\lambda_{\boldsymbol{m}'_n, \boldsymbol{m}_n}$$

where $\lambda_{\boldsymbol{m}'_n, \boldsymbol{m}_n}$ are some coefficients depending on $\boldsymbol{m}'_n$ and $\boldsymbol{m}_n$ only. Returning to formula (39) and denoting $(\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)' = (\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)\lambda_{\boldsymbol{m}'_n, \boldsymbol{m}_n}$ we have

$$V_n|\boldsymbol{m}_n, \xi_{n-1}; s\rangle = \sum_{\boldsymbol{m}'_n, s'} \big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big)(\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)'\, |\boldsymbol{m}'_n, \xi_{n-1}; s'\rangle. \tag{40}$$

Iterating the second formula in (36), we obtain the action formulas for $\{V_k\}$, $1 \leqslant k < n$. Thus, we deduce the following q-analogue of the Wigner–Eckart theorem.

Theorem 4. *If $\mathcal{V}$ is a Hilbert space and its GT basis $\{|\boldsymbol{m}_n, \xi_{n-1}; s\rangle\}$ is orthonormal, we have, for the components of vector operator $\{V_k\}$ on $\mathcal{V}$, the decomposition*

$$\langle \boldsymbol{m}'_n, \xi'_{n-1}; s'|V_k|\boldsymbol{m}_n, \xi_{n-1}; s\rangle = \big((\boldsymbol{m}'_n, \xi'_{n-1})|k, (\boldsymbol{m}_n, \xi_{n-1})\big)'(\boldsymbol{m}'_n, s' \| V \| \boldsymbol{m}_n, s)'$$

where

$$\begin{aligned}\big((\boldsymbol{m}'_n, \xi'_{n-1})|k, (\boldsymbol{m}_n, \xi_{n-1})\big)' = {} & \frac{\langle \boldsymbol{m}_{n+1}, \xi'_n|T_{\boldsymbol{m}_{n+1}}(I^+_{n+1,k})|\boldsymbol{m}_{n+1}, \xi_n\rangle}{\langle \boldsymbol{m}_{n+1}, \boldsymbol{m}'_n, \xi_{n-1}|T_{\boldsymbol{m}_{n+1}}(I_{n+1,n})|\boldsymbol{m}_{n+1}, \boldsymbol{m}_n, \xi_{n-1}\rangle} \\ & \times \big(n, (\boldsymbol{m}_n, \xi_{n-1})|(\boldsymbol{m}'_n, \xi_{n-1})\big) \qquad 1 \leqslant k < n\end{aligned}$$

(see comments after analogous formula (27)).

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References

[1] Drinfeld V G 1985 *Sov. Math.–Dokl.* **32** 254
[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[3] Gavrilik A M and Klimyk A U 1991 *Lett. Math. Phys.* **21** 215
[4] Fairlie D B 1990 *J. Phys. A: Math. Gen.* **23** L183
[5] Nelson J, Regge T and Zertuche F 1990 *Nucl. Phys.* B **339** 516
[6] Nelson J and Regge T 1991 *Phys. Lett.* B **272** 213
[7] Gavrilik A M 2000 *Proc. Inst. Math. (NAS Ukraine)* **30** 304
[8] Chekhov L O and Fock V V 2000 *Czech. J. Phys.* **50** 1201
[9] Gavrilik A M and Iorgov N Z 1997 *Methods Funct. Anal. Topology* **3** 51
[10] Iorgov N Z and Klimyk A U 2000 *Czech. J. Phys.* **50** 85
[11] Noumi M 1996 *Adv. Math.* **123** 16
[12] Noumi M, Umeda T and Wakayama M 1996 *Compos. Math.* **104** 227
[13] Iorgov N Z and Klimyk A U 2000 *Methods Funct. Anal. Topology* **6** 56
[14] Gavrilik A M 1973 Clebsch–Gordan coefficients for the direct product $[m_n] \otimes [1]$ of the group SO(n) representations *Institute for Theoretical Physics preprint* ITP-73-155P (in Russian)
[15] Klimyk A U 1979 *Matrix Elements and Clebsch–Gordan Coefficients of Group Representations* (Kiev: Naukova Dumka) (in Russian)
[16] Klimyk A and Schmüdgen K 1997 *Quantum Groups and Their Representations* (Berlin: Springer)
[17] Kachurik I I and Klimyk A U 2001 *J. Phys. A: Math. Gen.* **34** 793